Introduction to Integration

Table 1:

Some common integrals are:

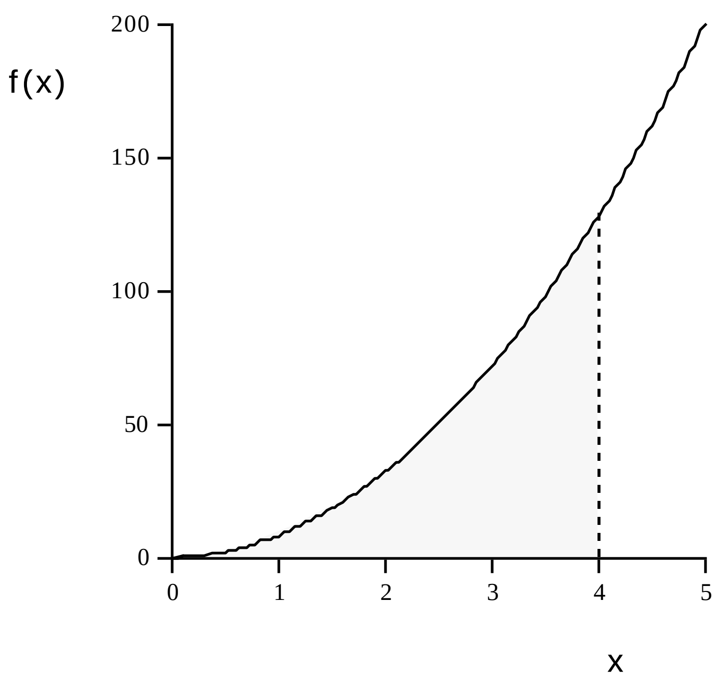
$$\int \frac{1}{x}\, dx = \ln x + C \qquad \int x^n\, dx = \frac{1}{n+1} x^{n+1} + C$$

$$\int e^x\, dx = e^x + C \qquad \int e^{cx}\, dx = \frac{1}{c} e^{cx} + C$$

$$\int \cos x\, dx = \sin x + C \qquad \int \sin x\, dx = -\cos x + C$$

$$\int \cosh x\, dx = \sinh x + C \qquad \int \sinh x\, dx = \cosh x + C$$

Figure 1: The integral of a function is a measure of the area between it and the x-axis. The case shown is $\int_0^4 f(x)\, dx$, where $f(x) = 8x^2$.



Integration is the inverse operation of differentiation and is a measure of the area under a curve. (Area below the x-axis counts as negative.)

How does one calculate an integral?

For example, let's calculate $\int 8x^2\,dx$. The 8 is a constant and, thus, can be taken outside the integral sign.

$$\int 8x^2\,dx = 8\int x^2\,dx = 8 \times \frac{1}{3}\,x^3 + C. \qquad (1)$$

This integral is called an indefinite integral because its value is not fully determined until the endpoints are specified. This ambiguity is dealt with by the addition of the constant C at the end. This constant of integration is added to the end because there are, in fact, an infinite number of solutions to the integral. To illustrate this, let's try differentiating two possible solutions, i.e. work backwards.

$d/dx\,((8/3)x^3 + 5)$	$d/dx((8/3)x^3 + 127)$
$= d/dx((8/3)x^3) + d/dx(5)$	$= d/dx((8/3)x^3) + d/dx(127)$
$= 8x^2 + 0$	$= 8x^2 + 0$
$= 8x^2$.	$= 8x^2$.

To calculate a definite integral and eliminate the constant of integration, we need limits for the integral. Suppose, in the example above, we wanted to find the area under the curve $8x^2$ from 0 to 4. This integral is represented as

$$\int_0^4 8x^2\,dx \qquad (2)$$

This is equal to $(8/3)\,4^3 - (8/3)\,0^3 = 512/3$.

One technique to try when you don't know how to calculate an integral is make a substitution in order to change the integral to something you do know. For example,

$$\int_0^4 \frac{1}{2+x}\,dx \qquad (3)$$

In this case, let u = 2 + x. Then, du = dx.

$$\int_2^6 \frac{1}{u}\, du \qquad (4)$$

The new limits are the limits for u; when x = 0, u = 2, and when x = 4, u = 6. This new integral for u is one we know; it is ln 6 - ln 2 = ln 3.

Another technique for calculating some integrals whose results are not obvious is a kind of inverse product rule and is called Integration by Parts.

$$\int u\, dv = uv - \int v\, du, \qquad (5)$$

where u and v are functions of x. As an example of this, let's do the integral $\int x \sin x\, dx$. Let u = x and dv = sinx dx. Then, du = dx and v = -cosx.

$$\int x \sin x\, dx = -x \cos x + \sin x. \qquad (6)$$

Another type of substitution is trig or hyperbolic substitution. Suppose that you have the integral

$$\int \frac{dx}{\sqrt{1 - x^2}} \qquad (7)$$

If you make the substitution $x = \sin\theta$ and $dx = \cos\theta\, d\theta$, then the integral becomes

$$\int \frac{\cos\theta\, d\theta}{\sqrt{1 - \sin^2\theta}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} \qquad (8)$$

$$\int d\theta = \theta + C = \sin^{-1}x + C \qquad (9)$$

Similar to this trig substitution is a hyperbolic substitution. Suppose that you had the integral

$$\int \frac{dx}{\sqrt{x^2 + 4}} \tag{10}$$

If you let x = 2 sinh θ and dx = 2 cosh θ dθ, then the integral becomes

$$\int \frac{2 \cosh d\theta}{2 \cosh \theta} = \sinh^{-1}\left(\frac{x}{2}\right) + C \tag{11}$$

Some Tricks for Exponential and Gaussian Integrals

Physicists frequently encounter integrals of the following form

$$\mathbf{I}_n^{(exp)}(\alpha) = \int_o^{\infty} x^n e^{-\alpha x} dx \tag{12a}$$

$$\mathbf{I}_n^{(gau)}(\alpha) = \int_o^{\infty} x^{2n} e^{-\alpha x^2} dx \tag{12b}$$

(In the latter case the integration range may run from –∞ to + ∞ which merely doubles the value.)

There are handy tricks for remembering how to do these integrations based on their respective values when n=0.

$$\mathbf{I}_o^{(exp)}(\alpha) = \int_o^{\infty} e^{-\alpha x} dx = -\frac{1}{\alpha} e^{-\alpha x}\Big|_o^{\infty}$$
$$= 1/\alpha \tag{13a}$$

$$\mathbf{I}_o^{(gau)}(\alpha) = \int_o^{\infty} e^{-\alpha x^2} dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$$

$$= \frac{1}{2} \sqrt{\left(\int_{-\infty}^{\infty} e^{-\alpha x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-\alpha y^2} dy\right)}$$

$$= \frac{1}{2} \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha (x^2+y^2)} dxdy}$$

Convert to polar coordinates taking advantage of the circular symmetry of the problem

$$dxdy \rightarrow 2\pi r dr$$

$$\mathbf{I}_o^{(gau)}(\alpha) = \frac{1}{2} \sqrt{2\pi \int_{o}^{\infty} r e^{-\alpha r^2} dr}$$

$$= \frac{1}{2} \sqrt{2\pi \left(\frac{-1}{2\alpha}\right) e^{-\alpha r^2} \Big|_o^{\infty}}$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{\alpha}} \tag{13b}$$

Next note that differentiations (under the integral sign) of $\mathbf{I}^{(exp)}$ and $\mathbf{I}^{(gau)}$ with respect to α bring down powers of x.

$$\frac{d}{d\alpha} \mathbf{I}_o^{(exp)}(\alpha) = \frac{d}{d\alpha} \int_{o}^{\infty} e^{-\alpha x} dx$$

$$= \int_{o}^{\infty} \frac{d}{d\alpha} e^{-\alpha x} dx$$

$$= -\int_{o}^{\infty} x\, e^{-\alpha x} dx \tag{14a}$$

$$= -I_1^{(exp)}$$

and

$$\frac{d}{d\alpha}\, I_o^{(gau)}(\alpha) = \frac{d}{d\alpha}\int_o^{\infty} e^{-\alpha x^2}\, dx$$

$$= \int_o^{\infty} \frac{d}{d\alpha}\, e^{-\alpha x^2}\, dx$$

$$= -\int_o^{\infty} x^2\, e^{-\alpha x^2}\, dx$$

$$= -I_1^{(gau)} \tag{14b}$$

Inverting egs. (14a,b) gives

$$I_1^{(exp)}(\alpha) = -\frac{d}{d\alpha}\, I_o^{(exp)}(\alpha)$$

$$= -\frac{d}{d\alpha}\left(\frac{1}{\alpha}\right)$$

$$= \frac{1}{\alpha^2} \tag{15a}$$

and

$$I_1^{(gau)}(\alpha) = -\frac{d}{d\alpha}\, I^{(gau)}(\alpha)$$

$$= -\frac{d}{d\alpha}\, \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}} \tag{15b}$$

If higher values of n are needed, further differentiation with respect to α will produce them similarly to egs. (14)

$$\frac{d}{d\alpha}\, I_n^{(exp)}(\alpha) = -I_{n+1}^{(exp)}(\alpha) \quad (16a)$$

$$\frac{d}{d\alpha}\, I_n^{(gau)}(\alpha) = -I_{n+1}^{(gau)}(\alpha) \qquad (16b)$$

Completing the square

One more trick is needed to handle integrals that are combinations of gaussians and exponentials. It is called *completing the square*. Consider

$$I^{(gau)} = \int_{-\infty}^{\infty} e^{-ax^2+bx}\, dx \qquad (17)$$

You'll see this one in quantum mechanics where the coefficient b can even be a complex number. (All of the following works whether b is real or complex.)

Consider the exponent

$$-a\left(x^2 - \frac{b}{a}\, x\right)$$

Note that by adding and substracting a constant we can put this expression in the form of a perfect square plus a constant

$$-a\left(x^2 - \frac{b}{a}\, x + \frac{1}{4}\,\frac{b^2}{a^2} - \frac{1}{4}\,\frac{b^2}{a^2}\right)$$

$$= -a\left(x - \frac{b}{2a}\right)^2 + \left(\frac{b^2}{4a}\right)$$

We put this into the integral on eq. (17) to find

$$I^{(gau)} = \int_{-\infty}^{\infty} e^{-a(x - \frac{b}{2a})^2} \, e^{\frac{b^2}{4a}} \, dx$$

Now we take the constant term out of the integrand and make the variable change $w = x - \frac{b}{2a}$ which alters neither the limits of integration or the differential element dx.

$$I^{(gau)} = e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} e^{-aw^2} \, dw$$

$$= \sqrt{\frac{\pi}{a}} \; e^{\frac{b^2}{4a}}$$

using eg. (13b).

If you need an integral other than one listed here, you should try looking in an integral table, the best of which is Gradshteyn and Ryzhik.

But, in order really to learn how to integrate (beyond what is shown here) one should take a course in complex analysis.

Reference: Hurley, *Intermediate Calculus*.